Two infinite nested roots.
https://www.linkedin.com/feed/update/urn:li:activity:6468373947728699392
2062.**Proposed by K**.**R**.**S**. **Sastry**, **Dodballapur**, **India**.
Find a positive integer n so that both the continued roots
$\sqrt{1995+\sqrt{n+\sqrt{1995+\sqrt{n+\ldots}}}}$
and
$\sqrt{n+\sqrt{1995+\sqrt{n+\sqrt{1995+\ldots}}}}$
converge to positive integers.
Solution by Arkady Alt, **San Jose**,**California**, **USA**.
Let a,b be real positive numbers.
Consider two sequences $(x_n),(y_n)$ defined by the system of recurrences
of the first order;

(**1**) $\begin{cases} x_{n+1} = \sqrt{a+y_n} \\ y_{n+1} = \sqrt{b+x_n} \end{cases}$ $n \in \mathbb{N}$ and $x_1 = \sqrt{a}, y_1 = \sqrt{b}$.

Let $h(x) := \sqrt{a+\sqrt{b+x}}$. Then $x_{n+2} = h(x_n), x_1 = \sqrt{a}, x_2 = \sqrt{a+\sqrt{b}}$
and $y_{n+2} = h(y_n)\ y_1 = \sqrt{b}, y_2 = \sqrt{b+\sqrt{a}}$
Both sequences are convergent and to prove this suffice consider one of them,
let it be(x_n).
Let $m := \max\{a,b\}$ and $m_n = \sqrt{m+\sqrt{m+\sqrt{m\ldots+\sqrt{m}}}}$ (n roots).
Since $x_n \leq m_n, n \in \mathbb{N}$ and $m_n \leq \dfrac{1+\sqrt{4m+1}}{2}$ then (x_n) is bounded from above.
Also, (x_n) is increasing sequence.
Indeed, since $x_1 < x_2 < x_3$ and for any $n \in \mathbb{N}$ assuming that
$x_{2n-1} < x_{2n} < x_{2n+1}, n \in \mathbb{N}$ we obtain
$h(x_{2n-1}) < h(x_{2n}) < h(x_{2n+1}) \Leftrightarrow x_{2n+1} < x_{2n+2} < x_{2n+3}$.
then by Math Induction $x_{2n-1} < x_{2n} < x_{2n+1}$ for any $n \in \mathbb{N}$, that is (x_n) is
increasing sequence.
As increasing and bounded from above (x_n) is convergent and (y_n)
is convergent by the same reason. Let $x := \lim_{n\to\infty} x_n$ and $y_n := \lim_{n\to\infty} y_n$.
Then passing to limit in (**1**) we obtain for (x,y) system of equation

(**2**) $\begin{cases} x = \sqrt{a+y} \\ y = \sqrt{b+x} \end{cases} \Leftrightarrow \begin{cases} x^2 = a+y \\ y^2 = b+x \end{cases}$.

Applying system (**4**) for nested roots of the problem we obtain

$\begin{cases} x^2 = 1995+y \\ y^2 = n+x \end{cases}$.

Let $y \in \mathbb{N}$ be such that $1995+y$ is a perfect square, that is $1995+y = (44+t)^2$.
Then $x = 44+t,\ y = x^2-1995 = (44+t)^2-1995 = t^2+88t-59$ and

$n = y^2 - x = (t^2 + 88t - 59)^2 - (44 + t) = t^4 + 176t^3 + 7626t^2 - 10\,385t + 3437$
for any $t \in \mathbb{N}$ (because $P(t) := t^4 + 176t^3 + 7626t^2 - 10\,385t + 3437 \geq 1$ for any $t \in \mathbb{N}$).
Thus, for any $t \in \mathbb{N}$ we have
$(x, y, n) = (44 + t, t^2 + 88t - 59, P(t))$
For example let $t = 1$ we obtain $x = 45, y = 84, n = P(t) = 855$